APPLICATIONS OF LAPLACE TRANSFORMATIONS TO SOLUTIONS OF DIFFERENTIAL EQUATIONS

APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

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Working rule to solve Differential Equation by Laplace Transform Method

Step 1. Take the Laplace Transform of both sides of the given Differential equation.

Step 2. Use the formula

$$L\{y'(t)\} = pL\{y(t)\} - y(0)$$

$$L\{y''(t)\} = p^2L\{y(t)\} - py(0) - y'(0)$$

Step 3.Replace $y(0), y'(0), y''(0)$ with the given initial conditions.

Step 4.After simplifying take the Inverse Laplace Transformation. This gives y(t) is the required solution of the given equation satisfying the given initial conditions.

1.Solve $y'' + y = 0$, $y = 1$, $Dy = 0$ when t=0 by using Laplace Transformation

Sol. . Given initial conditions $y(0) = 1, y'(0) = 0$

Given equation $y'' + y = 0$,

Taking Laplace Transformation on both sides $L\{y''\} + L\{y\} = 0$

$$=> p^2L\{y\} - py(0) - y'(0) + L\{y\} = 0$$

$$=> p^2L\{y\} - p(1) - 0 + L\{y\} = 0$$

$$=> L\{y\}(p^2 + 1) = p$$

$$=> L\{y\} = \frac{p}{p^2 + 1}$$

by taking Inverse Laplace Transformation on both sides, we get

$$y = L^{-1}\{\frac{p}{p^2 + 1}\}$$

$$=> y = cost$$

is the required solution

2. Solve $(D^2 - 2D + 2)y = 0$, if $y = Dy = 1$ when $t = 0$ by using laplace Transformation

Sol. Given initial conditions $y(0) = 1, y'(0) = 1$

Given equation $y'' - 2y' + 2y = 0$

Taking Inverse Laplace Transformation on both sides

$$p^2L\{y\} - py(0) - y'(0) - 2[pL\{y\} - y(0)] + 2L\{y\} = 0$$

$$=> \quad p^2L\{y\} - py(0) - y'(0) - 2[pL\{y\} - y(0)] + 2L\{y\} = 0$$

$$=> \quad p^2L\{y\} - p(1) - 1 - 2[pL\{y\} - 1] + 2L\{y\} = 0$$

$$=> \quad p^2L\{y\} - p - 1 - 2pL\{y\} + 2 + 2L\{y\} = 0$$

$$=> \quad p^2L\{y\} - p + 1 - 2pL\{y\} + 2L\{y\} = 0$$

$$=> \quad L\{y\}(p^2 - 2p + 2) = p - 1$$

$$=> \quad L\{y\} = \frac{p-1}{(p^2 - 2p + 2)}$$

$$\Rightarrow \qquad L\{y\} = \frac{p-1}{(p^2-2p+1)+1}$$

$$\Rightarrow \qquad L\{y\} = \frac{p-1}{(p-1)^2+1}$$

Taking Inverse Laplace Transformation on both sides

$$y = L^{-1}\{\frac{p-1}{(p-1)^2+1}\}$$

$$\Rightarrow \qquad y = e^t L^{-1}\{\frac{p}{p^2+1}\}$$

$$\Rightarrow \qquad y = e^t cost$$

3.Solve $(D^2 + D)y = t^2 + 2t, \ \ y(0) = 4, y'(0) = -2$

Sol: Given initial conditions $y(0) = 4, y'(0) = -2$

Given equation $y'' + y' = t^2 + 2t$

Taking L.T on both sides

$$L\{y''\} + L\{y'\} = L\{t^2\} + 2L\{t\}$$

$$=> \; p^2 L\{y\} - py(0) - y'(0) + \; pL\{y\} - y(0) = \frac{2}{p^3} + \frac{2}{p^2}$$

$$=> \; p^2 L\{y\} - 4p + 2 + pL\{y\} - 4 = \; \frac{2}{p^3} + \frac{2}{p^2}$$

$$=> L\{y\}(p^2 + p) = \frac{2}{p^3} + \frac{2}{p^2} + 4p + 2$$

$$=> L\{y\} = \frac{\frac{2}{p^3} + \frac{2}{p^2} + 4p + 2}{(p^2 + p)}$$

$$=> L\{y\} = \frac{\frac{2 + 2p + 4p^4 + 2p^3}{p^3}}{(p^2 + p)}$$

$$=> L\{y\} = \frac{2 + 2p + 4p^4 + 2p^3}{(p^2 + p)p^3}$$

$$=> L\{y\} = \frac{2 + 2p + 4p^4 + 2p^3}{p^4(p+1)}$$

$$=> L\{y\} = \frac{2(1+p) + 4p^4 \mp 2p^3}{p^4(p+1)}$$

$$=> L\{y\} = \frac{2}{p^4} + \frac{4}{p+1} + \frac{2}{p(p+1)}$$

$$=> L\{y\} = \frac{2}{p^4} + \frac{4}{p+1} + \frac{2}{p} - \frac{2}{p+1}$$

$$=> L\{y\} = \frac{2}{p^4} + \frac{2}{p} + \frac{2}{p+1}$$

Taking Inverse L.T. on both sides

$$y = L^{-1}\left\{\frac{2}{p^4} + \frac{2}{p} + \frac{2}{p+1}\right\}$$

$$y = L^{-1}\left\{\frac{2}{p^4}\right\} + L^{-1}\left\{\frac{2}{p}\right\} + L^{-1}\left\{\frac{2}{p+1}\right\}$$

$$y = 2\frac{t^3}{3!} + 2(1) + 2e^{-t}$$

Which is the required solution

4.Solve the equation $ty'' + (1-2t)y' - 2y = 0 \quad y(0) = 1, y'(0) = 2$

Sol. Given initial conditions $y(0) = 1, y'(0) = 2$

Given equation $ty'' + (1-2t)y' - 2y = 0$

Taking L.T. on both sides , we have

$$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$$

$$-\frac{d}{dp}(L\{y''\}) + pL\{y\} - y(0) + 2\frac{d}{dp}(L\{y'\}) - 2L\{y\} = 0$$

$$=> \; -\frac{d}{dp}\left(p^2L\{y\} - py(0) - y'(0)\right) + pL\{y\} - 1 + 2\frac{d}{dp}\left(pL\{y\} - y(0)\right) - 2L\{y\} = 0$$

$$=> \ -\frac{d}{dp}(p^2L\{y\} - p(1) - 2) + pL\{y\} - 1 + 2\frac{d}{dp}(pL\{y\} - 1) - 2L\{y\} = 0$$

$$=> \quad -\frac{d}{dp}(p^2L\{y\} - p - 2) + pL\{y\} - 1 + 2\frac{d}{dp}(pL\{y\} - 1) - 2L\{y\} = 0$$

Put $\quad L\{y\} = z$

$$=> \quad -\frac{d}{dp}(p^2z - p - 2) + pz - 1 + 2\frac{d}{dp}(pz - 1) - 2z = 0$$

$$=> \quad -\frac{d}{dp}(p^2z) + 1 - 0 + pz - 1 + 2\frac{d}{dp}(pz) - 0 - 2z = 0$$

$$=> \quad -2pz - p^2\frac{dz}{dp} + pz + 2z + 2p\frac{dz}{dp} - 2z = 0$$

$$\frac{dz}{dp}(-p^2 + 2p) - pz = 0$$

$$\frac{dz}{dp} = \frac{pz}{-p^2 + 2p}$$

$$\frac{dz}{dp} = \frac{-z}{p-2}$$

By Variable separable method

$$=> \frac{dz}{z} = \frac{-dp}{p-2}$$

$$=> \log z = -\log(p-2) + \log c$$

$$=> \log z = \log\left(\frac{c}{p-2}\right)$$

$$=> z = \frac{c}{p-2}$$

$$=> L\{y\} = \frac{c}{p-2}$$

Taking Inverse L.T. on both sides

$$y = cL^{-1}\left\{\frac{1}{(p-2)}\right\}$$

$$y = ce^{2t}$$ but y(0)=1

Therefore c=1

Hence $y = e^{2t}$ is the required solution.

THANK YOU